PROBLEMS AND QUESTIONS.

The solutions of problems are omitted this month to give more time for reports on problems proposed in the January issue. The following generalized puzzle is in keeping with the generalizing spirit of pure mathematics. It was suggested at a dinner of the Chicago Section of the American Mathematical Society.

A Puzzle Generalized.

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The original form of the puzzle is as follows: Given 6 counters in two like sets of 3 arranged alternately in a line. By moving pairs along the line without rotation to arrange in 3 moves the like sets together, leaving no gaps.

By using $+$, $-$ for the counters and 0's for the gaps the solution may be indicated thus.

$$\begin{array}{ccccccccc}
+ & - & + & - & + & - & & & \\
+ & - & + & 0 & 0 & - & - & + & \\
0 & 0 & + & + & - & - & - & + & \\
 & & 0 & 0 & - & - & - & + & + & +
\end{array}$$

The generalized problem substitutes m for 3 and $2m$ for 6, m being any integer.

The first proposition is that less than m moves is impossible. At first we have $2m - 1$ changes of sign and at the last 1. Hence $2m - 2$ changes are lost. Now no move can destroy more than 2 changes. For consider the moved pair and its former and latter neighbors. If the moved pair is like, not more than 2 changes exist at the start and so no more can be lost. If the moved pair is unlike, 3 changes may exist at the start but one is carried over. The first move cannot destroy more than one change and so m moves are necessary.

The solution is, however, for $m > 3$ not unique in its details, but can be described in its necessary features, and the latitude permitted in the way of variants indicated. The plan differs according to the residue of m modulo 4.

For $m = 4k$, arrange to count $2m + 2$ columns, the last two being occupied by zeros at the start, and for convenience cut off the last m columns by a bar

$$\begin{array}{cccc|cccccc}
1 & 2 & 3 & \cdots\ m+2 & m+3, & m+4, & \cdots & 2m, & 2m+1, & 2m+2 \\
+ & - & + & - & + & - & & - & 0 & 0
\end{array}$$

We describe a move by a substitution symbol. Thus $\begin{pmatrix} a, b \\ \alpha, \beta \end{pmatrix}$ means that the counters in the columns a, b are moved to the columns α, β.

The scheme of moves, in the first stage is now,

$$\binom{2 \quad\quad 3}{2m+1,\ 2m+2}\binom{m+5,\ m+6}{2 \quad\quad 3}\binom{6 \quad\quad 7}{m+5,\ m+6}\binom{m+9,\ m+10}{6 \quad\quad 7}\cdots$$
$$\binom{2m-3,\ 2m-2}{m-6,\ \ m-5}\binom{m-2,\ \ m-1}{2m-3,\ 2m-2}\binom{m+1,\ m+2}{m-2,\ m-1}.$$

The numbers on the left of the bar, are now

$$1,\ m+5,\ m+6,\ 4,\ 5,\ m+9,\ m+10,\ \cdots\ m+1,\ m+2,\ m,\ 0,\ 0.$$

Since odd numbers are indicated by $+$ and the even ones by $-$, they are seen to occur in like pairs. The numbers to the right are

$$m+3,\ m+4,\ 6,\ 7,\ m+7,\ m+8,\ 10,\ 11,\ \cdots\ m-2,\ m-1,\ 2m-1,\ 2m,\ 2,\ 3.$$

They begin and end with a $+$, the interior being occupied with alternate like pairs.

The number of moves taken so far is seen, by considering the successive pairs, to be

$$\frac{2(m-2-2)}{4}+2=\frac{m}{2}=2k.$$

There are now k pairs of $+$'s and k pairs of $-$'s on the left, and on the right $k-1$ pairs of $+$'s with k pairs of $-$'s, the extremes being also single $+$'s.

The second stage proceeds by moving a pair of $-$'s from right to left, and a pair of $+$'s from left to right to replace them, the only restriction on order being that the pair of $+$'s in the first two columns $(1,\ m+5)$ is to be moved last. This avoids breaking the rule concerning gaps. In this stage $2k$ moves are made, and in all $4k = m$ moves.

For $m = 4k+1$, the first part comprises $2k+1$ moves, the second $2k$ moves.

For $m = 4k+2$, there are $2k+1$ moves in each part.

For $m = 4k+3$, there are $2k+1$ moves in the first part, $2k+2$ in the second.

To save the lengthy symbolic representation it is sufficient to give the solutions for $m = 5, 6, 7$ as far as the end of the first stage.

If $m = 1$ the problem is already solved, for $m = 2$ it is impossible, and for $m = 3$ the solution does not conform to the general scheme; the reason being the lack of partners to carry over the marker which is left single at the end of the line.

$$m = 5$$

$$\begin{array}{l}
\left\{\begin{array}{ccccccc|ccccc}
+ & - & + & - & + & - & + & - & + & - & 0 & 0\\
+ & 0 & 0 & - & + & - & + & - & + & - & - & +\\
+ & + & - & - & + & - & 0 & 0 & + & - & - & +
\end{array}\right.\\
\quad\begin{array}{ccccccc|ccccc}
+ & + & - & 0 & 0 & - & - & + & + & - & - & +
\end{array}
\end{array}$$

$$m = 6$$

$$\begin{array}{l}
\left\{\begin{array}{cccccccc|cccccc}
+ & - & + & - & + & - & + & - & + & - & + & - & 0 & 0 \\
+ & 0 & 0 & - & + & - & + & - & + & - & + & - & - & +
\end{array}\right. \\
\left\{\begin{array}{cccccccc|cccccc}
+ & + & - & - & 0 & 0 & + & - & + & - & + & - & - & + \\
+ & + & - & - & - & + & + & - & + & 0 & 0 & - & - & +
\end{array}\right.
\end{array}$$

$$m = 7$$

$$\begin{array}{l}
\left\{\begin{array}{ccccccccc|ccccccc}
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & 0 & 0 \\
+ & 0 & 0 & - & + & - & + & - & + & - & + & - & + & - & - & + \\
+ & + & - & - & + & - & + & - & 0 & 0 & + & - & + & - & - & +
\end{array}\right. \\
\left\{\begin{array}{ccccccccc|ccccccc}
+ & + & - & - & + & 0 & 0 & - & - & + & + & - & + & - & - & + \\
+ & + & - & - & + & + & - & - & - & + & 0 & 0 & + & - & - & +
\end{array}\right.
\end{array}$$

for higher numbers the extra moves occur between the braces.

NOTES AND NEWS.

The Royal Society has awarded the Copley medal to Professor Felix Klein, of Göttingen, for his researches in mathematics.

A second, enlarged, edition of Professor W. F. Osgood's *Lehrbuch der Funktionentheorie* has appeared from the press of B. G. Teubner, of Leipzig.

Henry Holt and Company have just brought out a text on the *Theory and Practice of Mechanics* by S. E. Slocum, professor of applied mathematics in the University of Cincinnati.

Readers interested in studies of fundamental assumptions will enjoy a paper by A. B. Frizell on a set of 80 "Axioms of Ordinal Magnitudes," presented last August at the International Congress of Mathematicians.

A Peruvian knot record is described by Mr. L. Leland Locke in the *American Anthropologist*, April–June, 1912, p. 325. Apparently the knotted cords were used simply as numerical records. They seem to constitute "the earliest known decimal notation of the Western World, at any rate the earliest that admitted of easy transportation."

Professor Frederick Anderegg, of Oberlin College, will be absent on his sabbatical year during 1913–14. He expects to spend the time in Europe.

The Cambridge University Press has published a pamphlet of 51 pages by Sir Thomas L. Heath on "The Method of Archimedes," which was discovered by Heiberg in 1906 in Constantinople. The Greek manuscript in question is a tenth century copy of the works of Archimedes. Later the original writing was partially washed out and the parchment used for liturgical writing. In most places the original script is still legible. Among other works of Archimedes this manuscript contains "The Method" which had been thought irretrievably lost. It indicates the steps by which he worked his way to mathematical discoveries.